velocity of the unloading wave

$$a_1b^2 + 2a_0^2b - 3a_0^2a_1 = 0 \qquad (3.7)$$

Solving (3. 7) we find

$$b(0) = a_0\left[\left(\frac{a_0^2}{a_1^2} + 3\right)^{1/2} - \frac{a_0}{a_1}\right] \qquad (3.8)$$

Formula (3. 8) holds even in the absence of the delayed yielding effect, and is presented in [4].

The expressions obtained for the initial velocity of the unloading wave will be the starting point for constructing all unloading waves by the method of characteristics.

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STABILITY OF STEADY HELICAL MOTIONS OF A RIGID BODY IN A FLUID

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The problem of existence and stability of steady helical motions of a rigid body bounded by a simply connected syrface, was studied by Liapunov [1], using the Routh's theorem and its complement. Steklov in [2] established the existence of steady helical motions of a rigid body bounded by a multiply connected surface. Below we investigate the stability of the motions found by Steklov using the Routh's theorem and the Liapunov's complement, and we obtain the necessary conditions as well as some sufficient conditions of stability.

1. Let us suppose that a rigid body with several cavities filled with a perfect fluid, moves in an infinite, homogeneous, incompressible perfect fluid. We assume that the space occupied by the fluid (bounded by the surface of the body) and the cavities, are all multiply connected. We also assume that no forces act on the body and the fluid and that the motion of the fluid is irrotational. Taking any three mutually perpendicular straight lines rigidly connected to the body as the $OXYZ$-coordinate system, we shall denote the projections of the velocity of the origin on these axes by u, v and w and by p, q and r the projections of the angular velocity of the body. The principal rotations will be denoted by k_i and k_j' $(i = 1,\ldots, n)$, $(j = 1,\ldots, m)$.

Then, as shown by Steklov, the equations of motion are

$$\frac{d}{dt}\frac{\partial T}{\partial u} + \left(\frac{\partial T}{\partial w} + \beta_3\right) q - \left(\frac{\partial T}{\partial v} + \beta_2\right) r = 0 \qquad (1.1)$$

$$\frac{d}{dt}\frac{\partial T}{\partial p} + \left(\frac{\partial T}{\partial r} + \alpha_3\right) q - \left(\frac{\partial T}{\partial q} + \alpha_2\right) r + \left(\frac{\partial T}{\partial u} + \beta_1\right) v - \left(\frac{\partial T}{\partial v} + \beta_2\right) w = 0$$

$$(uvw,\ pqr,\ 123)$$

where T is the kinetic energy of the combined motion of the rigid body and the fluids, while α_i and β_i are certain constants dependent on the form of the body, on the cavities and on the cyclic motion of the fluid. Symbols $(uvw,\ pqr,\ 123)$ are permuted cyclically to yield the required equations. Equations (1.1) admit three first integrals

$$T = \text{const}, \qquad \left(\frac{\partial T}{\partial u} + \beta_1\right)^2 + \left(\frac{\partial T}{\partial v} + \beta_2\right)^2 + \left(\frac{\partial T}{\partial w} + \beta_3\right)^2 = \text{const}$$

$$\left(\frac{\partial T}{\partial u} + \beta_1\right)\left(\frac{\partial T}{\partial p} + \alpha_1\right) + \left(\frac{\partial T}{\partial v} + \beta_2\right)\left(\frac{\partial T}{\partial q} + \alpha_2\right) + \left(\frac{\partial T}{\partial w} + \beta_3\right)\left(\frac{\partial T}{\partial r} + \alpha_3\right) = \text{const}$$

Following [1], let us perform the following change of variables

$$\frac{\partial T}{\partial u} = x, \quad \frac{\partial T}{\partial v} = y, \quad \frac{\partial T}{\partial w} = z, \quad \frac{\partial T}{\partial p} = \xi, \quad \frac{\partial T}{\partial q} = \eta, \quad \frac{\partial T}{\partial r} = \zeta$$

Suitable choice of the coordinate system yields the following expression for T

$$2T = Sa_{11}x^2 + 2Sa_{12}xy + 2Sb_{11}x\xi + 2Sb_{12}(y\xi + z\eta) + Sc_1\xi^2 + C$$

$$a_{ij} = a_{ji},\ b_{ij} = b_{ji},\ c_i > 0 \quad (i = 1, 2, 3),\ (j = 1, 2, 3)\ (c_1 < c_2 < c_3)$$

Here and in the following, S denotes the summation of three terms obtained by simultaneous cyclic permutation $(xyz,\ \xi\eta\zeta,\ 123)$ of the terms under the S sign. Then the equations defining the steady helical motions, become

$$(b_{11} - \lambda)x + b_{12}y + b_{13}z + c_1\xi = \beta_1\lambda \qquad (1.2)$$

$$(a_{11} - \mu)x + a_{12}y + a_{13}z + (b_{11} - \lambda)\xi + b_{12}\eta + b_{13}\zeta = \mu\beta_1 + \lambda\alpha_1$$

$$(xyz,\ \xi\eta\zeta,\ 123)$$

Let us put, for brevity,

$$A_{ii} = a_{ii} - \frac{(b_{ii} - \lambda)^2}{c_i} - \frac{b_{ij}^2}{c_j} - \frac{b_{ik}^2}{c_k}, \quad A_{ij} = a_{ij} - \frac{b_{ki}b_{kj}}{c_k} - \frac{(b_{ii} - \lambda)b_{ij}}{c_i} - \frac{b_{ji}(b_{jj} - \lambda)}{c_j}$$

$$(i = 1, 2, 3;\ j = 1, 2, 3;\ k = 1, 2, 3;\ i \neq j,\ j \neq k,\ i \neq k)$$

Inserting ζ, η and ξ obtained from the first three equations of (1.2) into its last three equations, we obtain

$$(A_{11} - \mu)x + A_{12}y + A_{13}z = \Phi_1 \qquad (1.3)$$

$$(x\ y\ z,\ 1\ 2\ 3)$$

The necessary condition for (1.3) to have a solution is that

$$\det \| A_{ij} - \mu\delta_{ij} \| \neq 0, \qquad \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \qquad (i, j = 1, 2, 3) \qquad (1.4)$$

holds.

Using (1.3), we can obtain x, y and z and, consequently, ξ, η and ζ, for each value of μ satisfying (1.4). Thus we can obtain steady helical motions by assigning to λ any real values. Each value will have a corresponding infinite set of helical motions, whose axes will not coincide with the axes of the second order surface

$$SA_{11}x^2 + 2SA_{13}xz = \text{const}$$

2. We shall now investigate the stability of the steady helical motions relative to the x, y, z, ξ, η and ζ variables, assuming that in the perturbed motion we have

$$x' = x + \delta x,\ \xi' = \xi + \delta\xi, \qquad (xyz,\ \xi\eta\zeta)$$

Let us apply the Routh's theorem, assuming that

$$(x + \beta_1)^2 + (y + \beta_2)^2 + (z + \beta_3)^2 = h^2, \qquad S\,(x + \beta_1)(\xi + \alpha_1) = g \qquad (2.1)$$

and, that δx, δy, δz, $\delta\xi$, $\delta\eta$ and $\delta\zeta$ are the increments of x, y, z, ξ, η and ζ not affecting the constants h and g. We shall denote the corresponding increment in T, by δT.

$$2\delta T = S\,(A_{11} - \mu)(\delta x)^2 + 2SA_{23}\delta y\delta z + Sc_1(\delta\xi_0)^2 \qquad (2.2)$$

$$\delta\xi = \delta\xi_0 - \frac{b_{11} - \lambda}{c_1}\delta x - \frac{b_{12}}{c_1}\delta y - \frac{b_{13}}{c_1}\delta z \ (xyz,\ \xi\eta\zeta,\ 123)$$

In the new variables, conditions (2. 1) become

$$2S\,(x + \beta_1)\,\delta x + S\,(\delta x)^2 = 0$$

$$SX\delta x - S\,(x + \beta_1)\,\delta\xi_0 - S\delta x\delta\xi_0 + S\,\frac{b_{11} - \lambda}{c_1}(\delta x)^2 + S\left(\frac{b_{23}}{c_2} + \frac{b_{23}}{c_3}\right)\delta y\delta z = 0 \qquad (2.3)$$

where

$$X = 2\,(x + \beta_1)\frac{b_{11} - \lambda}{c_1} + (y + \beta_2)\left(\frac{b_{12}}{c_1} - \frac{b_{21}}{c_2}\right) + (z + \beta_3)\left(\frac{b_{13}}{c_1} - \frac{b_{13}}{c_3}\right) - f_1$$

$$f_1 = \alpha_1 + \frac{b_{11}\beta_1 + b_{12}\beta_2 + b_{13}\beta_3}{c_1} \ (XYZ,\ xyz,\ 123) \qquad (2.4)$$

Let us now assume that μ_1, μ_2 and μ_3 are the roots of the equation $\det\|A_{ij} - \mu\delta_{ij}\| = 0$. Motions with $\mu < \mu_1$ minimize T and are, consequently, stable at least with respect to the perturbations not affecting h and g.

Let us now consider the case $\mu > \mu_1$. Since the values δx, δy, δz, $\delta\xi$, $\delta\eta$, $\delta\zeta$ which concern us are vanishingly small, we can replace (2. 3) with

$$2S\,(x + \beta_1)\,\delta x = 0, \qquad SX\delta x - S\,(x + \beta_1)\,\delta\xi_0 = 0 \qquad (2.5)$$

To obtain the sufficient and necessary conditions for T to be minimum, we shall seek a minimum of the function (2. 2) under the conditions (2. 5) and

$$S\,(\delta x)^2 = C^2$$

The necessary condition for T to be minimum is, that $\min \delta T \geqslant 0$. On the other hand, if $\min \delta T = 0$ only when $C = 0$, then the conditions obtained will be sufficient. Seeking the minimum of T, we arrive at the following system of equations

$$(A_{11} - \mu - k)\,\delta x + A_{12}\delta y + A_{13}\delta z + (x + \beta_1)\,m + Xl = 0 \qquad (2.6)$$

$$c_1\delta\xi_0 - (x + \beta_1)\,l = 0 \qquad (xyz,\ \xi\eta\zeta,\ 123)$$

Inserting $\delta\xi_0$, $\delta\eta_0$ and $\delta\zeta_0$ from (2. 6) into the second equation of (2. 5), we obtain

$$SX\delta x - Hl = 0 \qquad \left(H = S\,\frac{(x + \beta_1)^2}{c_1}\right)$$

The system of five equations

$$(A_{11} - \mu - k)\,\delta x + A_{12}\delta y + A_{13}\delta z + (x + \beta_1)\,m + Xl = 0 \quad (xyz,\ 123)$$

$$S\,(x + \beta_1)\ \delta x = 0, \qquad SX\delta x - Hl = 0 \qquad (2.7)$$

is linear and homogeneous in δx, δy, δz, m and l, consequently k is given by

$$\begin{vmatrix} A_{11} - \mu - k & A_{12} & A_{13} & x + \beta_1 & X \\ A_{21} & A_{22} - \mu - k & A_{23} & y + \beta_2 & Y \\ A_{31} & A_{32} & A_{33} - \mu - k & z + \beta_3 & Z \\ x + \beta_1 & y + \beta_2 & z + \beta_3 & 0 & 0 \\ X & Y & Z & 0 & -H \end{vmatrix} = 0$$

whose roots are all real. Equations (2.6) and the relevant conditions, yield

$$2\delta T = kC^2$$

Minimum value of δT and the least root k, have the same sign and Eqs. (2.7) become

$$Hh^2k^2 - Pk + R = 0 \tag{2.8}$$

$$P = S\,[(y+\beta_2)Z - (z+\beta_3)Y]^2 + HS\,[A_{22} - \mu + A_{33} - \mu]\,(x+\beta_1)^2 -$$
$$-2HSA_{23}(y+\beta_2)\,(z+\beta_3)$$

$$R = S\,(A_{11}-\mu)\,[(y+\beta_2)Z - (z+\beta_3)Y]^2 + 2\,SA_{23}\,[(z+\beta_3)Y - (y+\beta_2)Z]^2 +$$
$$HS\,(A_{22}-\mu)(A_{33}-\mu) - A_{23}]\,(x+\beta_1)^2 + 2HS\,[A_{12}A_{13} - (A_{11}-\mu)\,A_{23}]\,(y+\beta_2)\,(z+\beta_3)$$

Conditions of positiveness of k are expressed by two inequalities $P > 0$ and $R > 0$.

We shall show that these conditions can be satisfied for $\lambda = 0$ and, when $\mu < \mu_3$. Let us write T in the following form:

$$2T = SA_{11}^*x^2 + 2SA_{23}^*yz + S\frac{p^2}{c_1}$$

where A_{ij}^* denote the values of the function A_{ij} when $\lambda = 0$; x, y and z depend, in this case, only on A_{ij} and μ.

Starting with any given values of x, y and z we can obtain any X, Y and Z by a suitable choice of the coefficients b_{ij} (at the same time the coefficients a_{ij} should be varied in such a manner, that A_{ij}^* and, consequently, x, y and z are not altered).

We shall choose the matrix (A_{ij}^*) to be positive definite. When $\mu > \mu_3$, the quadratic form of the unknowns $[(x+\beta_1)Y - (y+\beta_2)X]$ $\quad (xyz,\ XYZ,\ 123)$

entering the expression for R can be positive for some values of X, Y and Z, and we shall select these values as the initial ones. Increasing them proportionally, we can now make R positive, and continuing this process even further (if required) we can make P positive.

Thus we have shown that stable motions are possible under the perturbations not affecting h and g, when $\mu < \mu_3$. Sufficient conditions of such a stability are: $P > 0$ and $R > 0$.

If, amongst the motions corresponding to the values of h and g differing infinitesimally from those defined by the motion discussed above, a motion exists which differs from the latter by an infinitesimal amount, then we shall say that this motion varies continuously with h and g. Let us find the conditions of such continuity. Eqs. (1.3) yield the following differential equations:

$$(A_{11}-\mu)\,dx + A_{12}dy + A_{13}dz - (x+\beta_1)\,d\mu + Xd\lambda = 0 \qquad (xyz,\ 123) \tag{2.9}$$

In addition, we have

$$S\,(x+\beta_1)\,dx = 2hdh, \qquad S\,(\xi+\alpha_1)\,dx + S\,(x+\beta_1)\,d\xi = dg \tag{2.10}$$

On the other hand, if ξ, η, ζ, $d\xi$, $d\eta$ and $d\zeta$ in (2.10) are replaced by the corresponding expressions obtained by differentiating the first three equations of (1.2), we obtain

$$SX\,dx - Hd\lambda = -dg \tag{2.11}$$

Five Eqs. (2.9), (2.10) and (2.11) yield the values of dx, dy, dz, $d\mu$ and $d\lambda$ in terms of the given dg and dh, provided that the determinant of these equations does not vanish. Expanded form of this determinant is given by R (2.8), and the motions for which $R \neq 0$, are unconditionally stable.

Thus, using the Routh's theorem with the Liapunov's complement we have shown that unconditionally stable motions exist for $\mu < \mu_3$.

Sufficient conditions of stability are: $R > 0, \ P > 0$

3. Let us now write the equations of perturbed motion

$$\frac{d\delta x}{dt} = \delta y \frac{\partial T}{\partial \zeta} - \delta z \frac{\partial T}{\partial \eta} + (y + \delta y + \beta_2)\left(\frac{\partial T}{\partial \zeta}\right)_\delta - (z + \delta z + \beta_3)\left(\frac{\partial T}{\partial \eta}\right)_\delta$$

$$\frac{d\delta \xi}{dt} = \delta y \frac{\partial T}{\partial z} - \delta z \frac{\partial T}{\partial y} + (y + \delta y + \beta_2)\left(\frac{\partial T}{\partial z}\right)_\delta - (z + \delta z + \beta_3)\left(\frac{\partial T}{\partial y}\right)_\delta + \qquad (3.1)$$

$$+ \delta\eta \frac{\partial T}{\partial \zeta} - \delta\zeta \frac{\partial T}{\partial \eta} + (\eta + \delta\eta + \alpha_2)\left(\frac{\partial T}{\partial \zeta}\right)_\delta - (\zeta + \delta\zeta + \alpha_3)\left(\frac{\partial T}{\partial \eta}\right)_\delta \qquad (xyz,\ \xi\eta\zeta,\ 123)$$

Here the subsctipt δ denotes the result of replacing $x, y, \ldots$ in $\partial T/\partial x, \partial T/\partial y, \ldots$ with $\delta x, \delta y, \ldots$. Let us put

$$\frac{\partial \delta T}{\partial \delta \xi} = \omega_1 \qquad (\xi\eta\zeta,\ 123)$$

Then

$$2\delta T = 2A + S\frac{\omega_1^2}{c_1}, \qquad 2A = S(A_{11} - \mu)(\delta x)^2 + 2SA_{23}\delta y \delta z$$

Assuming $\sigma = S(b_{11} - \lambda)/c_1$, and following [1] we reduce system (3.1) to the form

$$\frac{d\delta x}{dt} = (y + \beta_2)\omega_3 - (z + \beta_3)\omega_2 \qquad (3.2)$$

$$\frac{1}{c_1}\frac{d\omega_1}{dt} = (y + \beta_2)\frac{\partial A}{\partial \delta z} - (z + \beta_3)\frac{\partial A}{\partial \delta y} + [Z - \sigma(z + \beta_3)]\omega_2 -$$

$$- [Y - \sigma(y + \beta_2)]\omega_3 \qquad (xyz,\ 123,\ XYZ)$$

We shall seek a particular solution of (3.2)

$$\delta x = \gamma_1 e^{kt}, \qquad \omega_1 = \theta_1 e^{kt} \qquad (xyz,\ 123) \qquad (\gamma_i, \theta_i = \text{const})$$

Let us put

$$2B = S(A_{11} - \mu)[(y + \beta_2)\theta_3 - (z + \beta_3)\theta_2]^2 +$$

$$2SA_{23}[(z + \beta_3)\theta_1 - (x + \beta_1)\theta_3][(x + \beta_1)\theta_2 - (y + \beta_2)\theta_1]$$

$$B_{11} = (A_{22} - \mu)(z + \beta_3)^2 + (A_{33} - \mu)(y + \beta_2)^2 - 2A_{23}(y + \beta_2)(z + \beta_3)$$

$$B_{23} = A_{12}(x + \beta_1)(z + \beta_3) - A_{13}(x + \beta_1)(y + \beta_2) -$$

$$-A_{23}(x + \beta_1)^2 - (A_{11} - \mu)(y + \beta_2)(z + \beta_3)$$

$$(x\ y\ z,\ 123)$$

On eliminating γ_1, γ_2 and γ_3, (3.2) becomes

$$\frac{\partial B}{\partial \theta_1} + \frac{k^2}{c_1}\theta_1 + [Y - \sigma(y + \beta_2)]k\theta_3 - [Z - \sigma(z + \beta_3)]k\theta_2 = 0 \qquad (xyz,\ XYZ,\ 123)$$

from which we eliminate θ_1, θ_2 and θ_3 to obtain the required equation

$$\det\|C_{ij}\| = 0 \quad (i, j = 1, 2, 3) \qquad C_{ii} = B_{ii} + \frac{k^2}{c_i} \qquad (3.3)$$

$$C_{12} = B_{12} - [Z - \sigma(z + \beta_3)]k, \qquad C_{23} = B_{23} - [X - \sigma(x + \beta_1)]k$$

$$C_{13} = B_{13} + [Y - \sigma(y + \beta_2)]k, \qquad C_{31} = B_{31} - [Y - \sigma(y + \beta_2)]k$$

$$C_{21} = B_{12} + [Z - \sigma(z + \beta_3)]k, \qquad C_{32} = B_{32} + [X - \sigma(x + \beta_1]k$$

which has two zero roots.

Dividing throughout by k^2, we obtain

$$\frac{k^4}{c_1 c_2 c_3} + Qk^2 + R = 0$$

$$Q = S\frac{1}{c_2c_3}B_{11} + S\frac{1}{c_1}[X - \sigma(x+\beta_1)]^2 \tag{3.4}$$

$$R = S\frac{1}{c_1}(B_{22}B_{33} - B_{23}^2) + SB_{11}[X - \sigma(x+\beta_1)]^2 + 2SB_{23}[Y - \sigma(y+\beta_2)][Z - \sigma(z+\beta_3)]$$

The quantity R appearing here is obtained from (2.8), while for Q we can have

$$Q = S\frac{1}{c_1}[X - \sigma(x+\beta_1)]^2 + HS\frac{A_{11}-\mu}{c_1} -$$

$$- S(A_{11}-\mu)\frac{(x+\beta_1)^2}{c_1^2} - 2SA_{23}\frac{(y+\beta_2)(z+\beta_3)}{c_2c_3}$$

The necessary condition of stability is that Eq.(3.3) has no roots with positive real parts, and this is equivalent to the following conditions:

$$Q^2 - \frac{4}{c_1c_2c_3}R \geqslant 0, \quad Q \geqslant 0, \quad R \geqslant 0 \tag{3.5}$$

These necessary conditions become sufficient for stability in the first approximation, if we neglect the equality signs in them. We shall now show that for sufficiently large λ^2, conditions (3.5) can always be satisfied when $\mu > \mu_3$. Let $\lambda \to \infty$. We then have

$$\frac{A_{ii}}{\lambda^2} = -\frac{1}{c_i} + O\left(\frac{1}{\lambda}\right), \quad \frac{A_{ij}}{\lambda^2} = O\left(\frac{1}{\lambda}\right), \quad \frac{\Phi_i}{\lambda^2} = \beta_i\left(-k + \frac{1}{c_i}\right) + O\left(\frac{1}{\lambda}\right)$$

$$(i = 1, 2, 3)\ \left(\lim_{\lambda\to\infty}\frac{\mu}{\lambda^2} = -k\right)\ k \neq \frac{1}{c_i}$$

and

$$\lim_{\lambda\to\infty}\frac{\mu_i}{\lambda^2} = -k_i = -\frac{1}{c_i}$$

Let us now vary μ with λ simultaneously and in such a manner, that

$$\lim_{\lambda\to\infty}\frac{\mu}{\lambda^2} = -\frac{1}{c_1} + \varepsilon \qquad (\varepsilon > 0)$$

where ε is a small number. Obviously, when λ is sufficiently large, then $\mu > \mu_3$.

Assuming that

$$x = -\beta_1 + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda^2}\right) \qquad (xyz,\ 123)$$

we find from Eqs.(1.2) dividing them by λ^2,

$$x = -\beta_1 + \frac{1}{k - \dfrac{1}{c_1}}\,\frac{1}{\lambda}\left(\alpha_1 + \beta_1\frac{b_{11}}{c_1} + \beta_2\frac{b_{12}}{c_1} + \beta_3\frac{b_{13}}{c_1}\right) + O\left(\frac{1}{\lambda^2}\right) \qquad (xyz,\ 123)$$

and hence

$$\lim_{\lambda\to\infty}\frac{\lambda(x+\beta_1)}{c_1} = \frac{1}{c_1k - 1}f_1$$

$$\lim_{\lambda\to\infty} X = -f_1\frac{c_1k+1}{c_1k-1} \qquad (xyz,\ XYZ\ 123)$$

Choosing ε sufficiently small, we have

$$\lim_{\lambda\to\infty} R = c_1\frac{1}{\varepsilon^4}f_1^4\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\left(\frac{1}{c_1} - \frac{1}{c_3}\right)$$

$$\lim_{\lambda\to\infty} Q = \frac{f_1^2}{c_1\varepsilon^2}\left[\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\left(\frac{1}{c_1} - \frac{1}{c_3}\right) + \frac{1}{c_2c_3}\right]$$

and we easily see that all stability conditions hold in the first approximation. Similarly, they can be satisfied when $\mu = \mu_1 + \varepsilon$. On the other hand, when $\mu = \mu_2 + \varepsilon$, then for sufficiently large λ and sufficiently small ε, the necessary conditions of stability are violated.

Thus we have shown that, when $\mu > \mu_3$, then motions exist which are stable in the first approximation, while when μ is almost equal to μ_2, then the motions are unstable.

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ON THE ANALYSIS OF RESONANCES IN NONLINEAR SYSTEMS

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Perturbed resonant solutions of an essentially nonlinear real system containing two phases and a quasiconstant vector, are constructed over an infinite time interval. The first Liapunov method and well known Weierstrass theorems on implicit functions are used to derive the sufficient conditions of stability of perturbed resonant motions. The results obtained are interesting and may find application to certain problems of the theory of nonlinear oscillations.

1. Statement of the problem. We investigate, in the resonant region, a perturbed system of $(l+2)$ equations of the form

$$da / dt = \varepsilon A(\theta, a, \psi, \varepsilon)$$

$$d\psi / dt = \Omega(a) + \varepsilon \Psi(\theta, a, \psi, \varepsilon), \qquad d\theta / dt = \sigma(a) + \varepsilon N(\theta, a, \psi, \varepsilon) \qquad (1.1)$$

Here $t \in [t_0, \infty)$ is time, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ is a small parameter, a is a quasiconstant l-dimensional vector $(|a - a_0^*| < \delta)$, while ψ and θ denote scalar phases $(|\psi|, |\theta| < \infty)$. We assume that the functions A, Ω, Ψ, σ and N are sufficiently smooth in all their arguments within the indicated region and are periodic in θ and ψ, the periods being equal to $2\pi / \nu$ and 2π, respectively. The degree of smoothness shall be established below. We also assume that at least one of the phases, say θ, is rotating, i.e. $\sigma(a) > 0$.

We construct solutions of (1.1) and investigate their Liapunov stability. These solutions are such that when $\varepsilon = 0$ then they have the form

$$a_0, \quad \psi_0 = \Omega(a_0)(t - t_0) + \alpha, \quad \theta_0 = \sigma(a_0)(t - t_0) + \beta$$

while for $\varepsilon \neq 0$ they do not differ appreciably from the above magnitudes for all real t (in the above expressions a_0, α and β are certain constants). In the present paper we study the resonant case, when

$$m\Omega(a_0) = n\nu\sigma(a_0) \qquad (1.2)$$

where m and n are "not very large" integers [1] and n may become equal to zero, i.e. the phases ψ may be oscillating.

System (1.1) appears in many problems of the theory of nonlinear oscillations and in particular, in the problems on forced motions in a system with one degree of freedom and little varying parameters, in certain strongly connected autonomous systems, e. a.

Similar systems were investigated in [2 and 3] using the concept of averaging over the interval $\Delta t \sim 1 / \sqrt{\varepsilon}$. A particular case of the system (1.1) $(l = 1, \theta \equiv \nu t)$ was investigated by the author, who considered its general and particular solution [4 and 5] over the interval $t \in [t_0, \infty)$.